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Some Oscillation Criteria for Difference Equations

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Abstract—Consider the difference equation $x_{n+1} - x_n + p_n x_{n-k} = 0$, where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer. It is proved that all solutions oscillate if there exists some positive integer l such that

$$\limsup_{n\to\infty} \left\{ \sum_{i=0}^{k} p_{n-i} + \prod_{i=0}^{k} \sum_{j=1}^{k} p_{n-i+j} + \sum_{m=0}^{l-1} \sum_{i=1}^{k} \prod_{j=0}^{m+1} p_{n-jk-i} \right\} > 1$$

or

$$\limsup_{n\to\infty} \left\{ \sum_{i=1}^{k} p_{n-i} + \prod_{i=1}^{k} \sum_{j=1}^{k} p_{n-i+j} + \sum_{m=0}^{l-1} \sum_{i=1}^{k} \prod_{j=0}^{m+1} p_{n-jk-i} \right\} > 1.$$

Keywords—Oscillation, Nonoscillation, Difference equation.

1. INTRODUCTION

The problem of establishing sufficient conditions for the oscillation of all solutions of the difference equation

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \qquad n = 0, 1, 2, \dots, \tag{1}$$

where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer, has been the subject of many recent investigations. See, for example, [1–10] and the references cited therein.

By a solution of equation (1) we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and which satisfies equation (1) for $n \geq 0$. A solution $\{x_n\}$ of equation (1) is said to be *oscillatory* if the terms x_n of the solution are not eventually positive or eventually negative. Otherwise, the solution is called *nonoscillatory*.

Erbe and Zhang [2] first proved that all solutions of equation (1) oscillate if

$$\limsup_{n\to\infty} \sum_{i=0}^{k} p_{n-i} > 1 \tag{2}$$

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or

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}}. \tag{3}$$

Later, condition (3) was improved by Ladas, Philos and Sficas [6], to

$$\liminf_{n\to\infty} \sum_{i=1}^{k} p_{n-i} > \left(\frac{k}{k+1}\right)^{k+1}. \tag{4}$$

Condition (2) and (4) have been extensively exploited in the study of the oscillation of various difference equations. See, for example, [3,4,6] and the references cited therein.

There are several papers related to the improvement of condition (4). See [8–10]. In a different direction, Stavroulakis [7] considered the case when none of conditions (2) and (4) is satisfied and proved that all solutions of equation (1) oscillate if

$$\liminf_{n\to\infty} \sum_{i=1}^{k} p_{n-i} > M > 0$$

and

$$\limsup_{n\to\infty} p_n > 1 - \left(\frac{M}{2}\right)^2. \tag{5}$$

In this paper, new sufficient conditions for the oscillation of all solutions of equation (1) are established. These conditions concern the case when none of conditions (2)–(5) is satisfied. As an application of our results to a class of differential equations with piecewise constant argument considered by Aftabizadeh, Wiener and Xu [11], a sufficient condition in [11] is improved.

In the following, for convenience, we will assume that inequalities about values of sequences are satisfied eventually for all large n.

2. MAIN RESULTS

THEOREM 1. *Assume that there exists some positive integer l such that*

$$\limsup_{n\to\infty} \left\{ \sum_{i=0}^{k} p_{n-i} + \prod_{i=0}^{k} \sum_{j=1}^{k} p_{n-i+j} + \sum_{m=0}^{l-1} \sum_{i=1}^{k} \prod_{j=0}^{m+1} p_{n-jk-i} \right\} > 1. \tag{6}$$

Then all solutions of equation (1) oscillate.

PROOF. Suppose to the contrary that equation (1) has an eventually positive solution $\{x_n\}$. By (1), we have

$$x_{n-i} = x_{n-i+1} + p_{n-i} x_{n-k-i}, \qquad i = 1, 2, \dots, k.$$

Summing both sides of the above equality from $i = 1$ to $i = k$ yields

$$x_{n-k} = x_n + \sum_{i=1}^{k} p_{n-i} x_{n-k-i}. \tag{7}$$

From (1), for any positive integer j, we have

$$x_{n-k-j} = x_{n-k-j+1} + p_{n-k-j} x_{n-k-j-k}. \tag{8}$$

Substituting (8) for $j = i$ into (7), we have

$$x_{n-k} = x_n + \sum_{i=1}^{k} p_{n-i} x_{n-k-i+1} + \sum_{i=1}^{k} p_{n-i} p_{n-k-i} x_{n-i-2k}.$$

Substituting (8) for $j = i + k$ into the last equality, we have

$$x_{n-k} = x_n + \sum_{i=1}^{k} p_{n-i}x_{n-k-i+1} + \sum_{i=1}^{k} p_{n-i}p_{n-k-i}x_{n-2k-i+1} + \sum_{i=1}^{k} p_{n-i}p_{n-k-i}p_{n-2k-i}x_{n-i-3k}.$$

Following this iterative procedure, by induction, we obtain

$$\begin{aligned} x_{n-k} = x_n &+ \sum_{i=1}^{k} p_{n-i}x_{n-k-i+1} + \sum_{i=1}^{k} p_{n-i}p_{n-k-i}x_{n-2k-i+1} \\ &+ \sum_{i=1}^{k} p_{n-i}p_{n-k-i}p_{n-2k-i}x_{n-3k-i+1} + \cdots \\ &+ \sum_{i=1}^{k} p_{n-i}p_{n-k-i}\cdots p_{n-lk-i}x_{n-(l+1)k-i+1} \\ &+ \sum_{i=1}^{k} p_{n-i}p_{n-k-i}\cdots p_{n-(l+1)k-i}x_{n-i-(l+2)k}. \end{aligned}$$

Removing the last term of the last equality, we have

$$x_{n-k} \geq x_n + \sum_{i=1}^{k} p_{n-i}x_{n-k-i+1} + \sum_{m=0}^{l-1} \sum_{i=1}^{k} x_{n-(m+2)k-i+1} \prod_{j=0}^{m+1} p_{n-jk-i}. \tag{9}$$

From (1), we have

$$x_{n+j+1} - x_{n+j} + p_{n+j}x_{n+j-k} = 0, \qquad j = 0, 1, 2, \ldots, k-1.$$

Summing both sides of the last equality from $j = 0$ to $j = k - 1$, we have

$$x_n = x_{n+k} + \sum_{j=0}^{k-1} p_{n+j}x_{n+j-k}. \tag{10}$$

Since $\{x_n\}$ is eventually decreasing, it follows that

$$x_n > \sum_{j=0}^{k-1} p_{n+j}x_{n+j-k} \geq \left(\sum_{j=0}^{k-1} p_{n+j}\right) x_{n-1}, \tag{11}$$

and so

$$x_{n+1} > \left(\sum_{j=1}^{k} p_{n+j}\right) x_n > \left(\prod_{i=0}^{k} \sum_{j=1}^{k} p_{n-i+j}\right) x_{n-k}. \tag{12}$$

From (1), we have

$$x_n = x_{n+1} + p_n x_{n-k}. \tag{13}$$

Substituting (13) into (9), then using (12) and the fact that $\{x_n\}$ is eventually decreasing, we have

$$\begin{aligned} x_{n-k} &\geq \left(\prod_{i=0}^{k} \sum_{j=1}^{k} p_{n-i+j}\right) x_{n-k} + p_n x_{n-k} + \sum_{i=1}^{k} p_{n-i}x_{n-k} + \sum_{m=0}^{l-1} \sum_{i=1}^{k} x_{n-(m+2)k} \prod_{j=0}^{m+1} p_{n-jk-i} \\ &= \left(\sum_{i=0}^{k} p_{n-i}\right) x_{n-k} + \left(\prod_{i=0}^{k} \sum_{j=1}^{k} p_{n-i+j}\right) x_{n-k} + \sum_{m=0}^{l-1} x_{n-(m+2)k} \sum_{i=1}^{k} \prod_{j=0}^{m+1} p_{n-jk-i}. \end{aligned}$$

Substituting $x_{n-(m+2)k} \geq x_{n-k}$ into the last inequality, we obtain

$$x_{n-k} \geq \left(\sum_{i=0}^{k} p_{n-i} + \prod_{i=0}^{k} \sum_{j=1}^{k} p_{n-i+j} + \sum_{m=0}^{l-1} \sum_{i=1}^{k} \prod_{j=0}^{m+1} p_{n-jk-i}\right) x_{n-k}.$$

Dividing both sides of the last inequality by x_{n-k}, then taking the limit superior as $n \to \infty$, we are led to a contradiction. The proof is complete.

THEOREM 2. *Assume that there exists some positive integer l such that*

$$\limsup_{n\to\infty}\left\{\sum_{i=1}^{k} p_{n-i}+\prod_{i=1}^{k}\sum_{j=1}^{k} p_{n-i+j}+\sum_{m=0}^{l-1}\sum_{i=1}^{k}\prod_{j=0}^{m+1} p_{n-jk-i}\right\}>1. \tag{14}$$

Then all solutions of equation (1) oscillate.

PROOF. As in the proof of Theorem 1, we have that (9) and (11) hold. From (11), we have

$$x_n>\left(\prod_{i=1}^{k}\sum_{j=1}^{k} p_{n-i+j}\right)x_{n-k}. \tag{15}$$

Substituting (15) into (9) and using the fact that $\{x_n\}$ is eventually decreasing, we obtain

$$x_{n-k}\ge\left(\sum_{i=1}^{k} p_{n-i}+\prod_{i=1}^{k}\sum_{j=1}^{k} p_{n-i+j}+\sum_{m=0}^{l-1}\sum_{i=1}^{k}\prod_{j=0}^{m+1} p_{n-jk-i}\right)x_{n-k}. \tag{16}$$

Dividing both sides of the last inequality by x_{n-k}, and then taking the limit superior as $n\to\infty$, we are led to a contradiction with (14). The proof is complete.

REMARK 1. It should be noted that condition (6) is an obvious improvement of condition (2). On the other hand, it should be emphasized that condition (14) is different from condition (6) when $k>1$.

From Theorem 1 and Theorem 2, we derive the following corollaries.

COROLLARY 1. *Assume that*

$$\limsup_{n\to\infty}\sum_{i=0}^{k} p_{n-i}>1-\alpha^{k+1}-\frac{k\beta^2}{1-\beta}, \tag{17}$$

where

$$\alpha=\liminf_{n\to\infty}\sum_{i=1}^{k} p_{n-i},\qquad \beta=\liminf_{n\to\infty} p_n.$$

Then all solutions of equation (1) oscillate.

COROLLARY 2. *Let α and β be as in Corollary 1. Assume that*

$$\limsup_{n\to\infty}\sum_{i=1}^{k} p_{n-i}>1-\alpha^{k}-\frac{k\beta^2}{1-\beta}, \tag{18}$$

Then all solutions of equation (1) oscillate.

We give only the proof of Corollary 1. The proof of Corollary 2 can be given similarly and is omitted.

PROOF OF COROLLARY 1. If $\alpha>k^{k+1}/(k+1)^{k+1}$, then, by (4), all solutions of (1) oscillate. Observe that $\beta\le\alpha/k$. Thus, we will assume that $0\le\alpha/k\le k^{k+1}/(k+1)^{k+1}$. If $\beta=\alpha=0$, by (2), all solutions of (1) oscillate. If $\beta=0,\alpha>0$, then condition (17) reduces to

$$\limsup_{n\to\infty}\sum_{i=0}^{k} p_{n-i}>1-\alpha^{k+1}. \tag{19}$$

For some sufficiently small $\varepsilon \in (0,\alpha)$ we have $\sum_{i=1}^{k} p_{n-i} \geq \alpha - \varepsilon$ and

$$\limsup_{n\to\infty} \sum_{i=0}^{k} p_{n-i} > 1 - (\alpha - \varepsilon)^{k+1}. \tag{20}$$

Thus, we have

$$\prod_{i=0}^{k} \sum_{j=1}^{k} p_{n-i+j} \geq (\alpha - \varepsilon)^{k+1}.$$

This, in view of (20), implies that (6) holds. By Theorem 1, all solutions of equation (1) oscillate. We now consider the case when $\beta > 0$. By Theorem 1, it suffices to prove that condition (17) implies condition (6). From (17), it follows that, for some sufficiently small $\varepsilon \in (0,\beta)$ we have

$$\limsup_{n\to\infty} \sum_{i=0}^{k} p_{n-i} > 1 - (\alpha - \varepsilon)^{k+1} - \frac{k(\beta-\varepsilon)^2}{1-(\beta-\varepsilon)}. \tag{21}$$

The last inequality, in view of the fact that $(\beta - \varepsilon)^m \to 0$ as $m \to \infty$, implies that for some sufficiently large positive integer l

$$\begin{aligned}\limsup_{n\to\infty} \sum_{i=0}^{k} p_{n-i} &> 1 - (\alpha-\varepsilon)^{k+1} - \frac{k(\beta-\varepsilon)^2\left\{1-(\beta-\varepsilon)^l\right\}}{1-(\beta-\varepsilon)} \\ &= 1 - (\alpha-\varepsilon)^{k+1} - k(\beta-\varepsilon)^2\left\{1 + (\beta-\varepsilon) + (\beta-\varepsilon)^2 + \cdots + (\beta-\varepsilon)^{l-1}\right\}.\end{aligned}$$

This leads to (6) because

$$\prod_{i=0}^{k} \sum_{j=1}^{k} p_{n-i+j} + \sum_{m=0}^{l-1} \sum_{i=1}^{k} \prod_{j=0}^{m+1} p_{n-jk-i} \geq (\alpha-\varepsilon)^{k+1} + k(\beta-\varepsilon)^2 + \cdots + k(\beta-\varepsilon)^{l+1}.$$

The proof is complete.

EXAMPLE. Consider the equation

$$x_{n+1} - x_n + p_n x_{n-2} = 0, \qquad n = 0,1,2,\ldots, \tag{22}$$

where

$$p_{2n} = \frac{1}{9}, \quad p_{2n+1} = \frac{1}{9} + \frac{53}{81}\sin^2\frac{n\pi}{2}, \qquad n = 0,1,2,\ldots.$$

It is easy to see that

$$\beta = \liminf_{n\to\infty} p_n = \frac{1}{9}, \qquad \alpha = \liminf_{n\to\infty} \sum_{i=1}^{2} p_{n-i} = \frac{2}{9},$$

$$\limsup_{n\to\infty} \sum_{i=0}^{2} p_{n-i} = \frac{1}{3} + \frac{53}{81} < 1, \qquad \limsup_{n\to\infty} p_n = \frac{1}{9} + \frac{53}{81} < 1 - \frac{\alpha^2}{4}.$$

Thus, none of the conditions (2)–(5) is satisfied. However, it is easy to see that

$$\limsup_{n\to\infty} \sum_{i=0}^{2} p_{n-i} > 1 - \alpha^3 - \frac{2\beta^2}{1-\beta},$$

and so (17) is satisfied. By Corollary 1, all solutions of (22) oscillate.

In the following, we consider an application of our results to the differential equations with piecewise constant arguments of the form

$$y'(t) + a(t)y(t) + b(t)y([t-1]) = 0, \qquad t \geq 0, \tag{23}$$

where $a(t)$ and $b(t)$ are continuous functions on $[-1,\infty), b(t) \geq 0$ for $t \geq 0$, and $[\cdot]$ denotes the greatest integer function.

As is customary, a nontrivial solution of equation (23) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory.

In [11], Aftabizadeh, Wiener and Xu studied the oscillation of equation (23) and proved that all solutions oscillate if

$$\limsup_{n\to\infty} \int_{n-1}^{n} b(t) \exp\left(\int_{n-2}^{t} a(s)\,ds\right) dt > 1. \tag{24}$$

By appling Theorem 2 to equation (23), we can establish the following result which improves condition (24).

COROLLARY 3. *Assume that there exists some positive integer l such that*

$$\limsup_{n\to\infty} \left\{ \int_{n}^{n+1} b(t) \exp\left(\int_{n-1}^{t} a(s)\,ds\right) dt + \sum_{m=0}^{l-1} \prod_{j=0}^{m} \int_{n-j-1}^{n-j} b(t) \exp\left(\int_{n-j-2}^{t} a(s)\,ds\right) dt \right\} > 1. \tag{25}$$

Then all solutions of equation (23) oscillate.

PROOF. Assume that equation (23) has an eventually positive solution $y(t)$. Then, as in [11], for $n = 0,1,2,\dots$, in the interval $n \leq t < n+1$, $y(t)$ satisfies

$$y'(t) + a(t)y(t) + b(t)A_{n-1} = 0, \qquad n \leq t < n+1, \tag{26}$$

where we use the notation $A_n = y(n)$ for $n = -1,0,1,\dots$. Equation (26) can be rewritten as

$$\left(y(t)\exp\left(\int_n^t a(s)\,ds\right)\right)' + b(t)\exp\left(\int_n^t a(s)\,ds\right) A_{n-1} = 0, \qquad n \leq t < n+1.$$

Taking integral from n to $t, n \leq t < n+1$, we have

$$y(t)\exp\left(\int_n^t a(u)\,du\right) - A_n + \left(\int_n^t b(s)\exp\left(\int_n^s a(u)\,du\right) ds\right) A_{n-1} = 0. \tag{27}$$

Set for $n = 0,1,\dots$

$$P_n = \exp\left(\int_n^{n+1} a(t)\,dt\right), \qquad Q_n = \int_n^{n+1} b(t)\exp\left(\int_n^t a(s)\,ds\right) dt. \tag{28}$$

In (27), letting $t \to n+1$ and by continuity (cf. [11]), we have

$$P_n A_{n+1} - A_n + Q_n A_{n-1} = 0, \qquad n = 0,1,2,\dots. \tag{29}$$

Set $B_n = A_n \prod_{j=1}^{n-1} P_j$ for $n > 1$. Then, $\{B_n\}$ is eventually positive and satisfies

$$B_{n+1} - B_n + Q_n P_{n-1} B_{n-1} = 0. \tag{30}$$

On the other hand, by Theorem 2, we see that condition (25) implies that all solutions of the difference equation

$$x_{n+1} - x_n + Q_n P_{n-1} x_{n-1} = 0$$

oscillate. This contradiction completes the proof.

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